ON THE STRUCTURE OF GAME PROBLEMS OF DYNAMICS

PMM Vol. 35, №1, 1971, pp. 110-122
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(Received August 17. 1970)

The construction of optimal player strategies defining situations of the saddle-point type in certain differential games is described. The approach used to investigate the differential games under consideration is based on [1], where we introduced the notions of mixed player strategies and provided an alternative which holds for this class of strategies. The contents of the present paper are related to those of [2-5].

1. Let the motion of a controlled system be described by an equation of the form

$$dx / dt = f(t, x, u, v), \qquad x[t_0] = x_0 \tag{1.1}$$

Here x is an n-dimensional phase vector and $f(t, x, u, v)$ is a continuous vector function which satisfies Lipschitz' condition in x; u and v are the vector controlling forces at the disposal of the first and second player, respectively. We assume that the choice of controls u and v is subject to restrictions of the form

$$u \in P, \qquad v \in Q \tag{1.2}$$

where P and Q are closed bounded sets in the corresponding vector spaces.

The games problems (differential games) to be considered in the present paper are as follows.

Problem 1.1. Let some closed sets N and G be given in the vector space $p = \{t, x\}$. The first player who chooses the control u seeks to effect system motion (1.1) in such a way that for any permissible behavior of his opponent the point $p[t] = \{t, x[t]\}$ arrives at the set N in the shortest time, and, moreover, the phase restriction $p[t] \in G$ holds throughout the time that the point $p[t]$ travels from the initial state $p_0 = \{t_0, x_0\}$ to the set N. The second player strives to achieve the opposite aim. In other words, he seeks to effect a system motion (1.1) in such a way that either the phase restriction $p[t] \in G$ is violated before the point $p[t]$ arrives at the set N, or the point $p[t]$ does not reach the set N within the maximum possible time interval $[t_0, \vartheta]$. The payoff in this differential game is defined as follows:

$$\gamma = \begin{cases} \vartheta(x[\cdot]) - t_0 & \text{if} \quad p[t] = \{t, x[t]\} \in G, \quad t_0 \leqslant t \leqslant \vartheta(x[\cdot]) \\ \infty & \text{if} \quad p[t_*] \notin G \quad \text{for some} \quad t_* \in [t_0, \vartheta(x[\cdot])] \end{cases} \tag{1.3}$$

Here $\vartheta(x[\cdot])$ denotes the instant when the inclusion $\{t, x[t]\} \in N$ first occurs for some motion $x[t]$; when the point $p[t] = \{t, x[t]\}$ does not arrive at N for $t \geqslant t_0$ we set $\vartheta(x[\cdot]) = \infty$.

In stating this problem we assume that each of the players knows the realized game position $p[t] = \{t, x[t]\}$ at each instant $t \geqslant t_0$, but does not know the control chosen by his opponent at this instant and subsequent instants. We shall seek the solution of this problem (which consists in determining the strategies yielding a saddle point) in the class of mixed strategies introduced in [1].

Problem 1.2. Let the motion of the controlled system be described by Eq. (1.1), and let some closed set N be specified in the vector space $p = \{t, x\}$. The payoff is defined by the equation

$$\gamma = \max_t \varphi(t, x[t]) \quad \text{for} \quad t_0 \leqslant t \leqslant \min\{T, \vartheta(x[\cdot])\} \tag{1.4}$$

where $\varphi(t, x)$ is a given continuous function, T is some finite instant which limits the duration of the game, and $\vartheta(x[\cdot])$, as above, is the instant when the point $p[t]$ first reaches N. We assume that the first player seeks to minimize γ (1.4); the second player, on the contrary, seeks to maximize this quantity. The character of the information to which the players have access is the same as in the previous differential game.

As above, the problem consists in determining the player strategies which produce a saddle point.

Problem 1.3. This problem differs from the previous one only in the fact that the payoff is defined by the relation

$$\gamma = \begin{cases} \max_t \varphi(t, x) & \text{for } t_0 \leqslant t \leqslant \vartheta(x[\cdot]), \quad \text{if } \vartheta(x[\cdot]) \leqslant T \\ \infty & \text{if } \vartheta(x[\cdot]) > T \end{cases} \tag{1.5}$$

In this game the first player must bring the point $p[t]$ to the set N not later than at the instant $t = T$; otherwise he loses the game.

Problem 1.4. The payoff in our last differential game is defined by the equation

$$\gamma = \varphi(\vartheta, x[\vartheta]) + \int_{t_0}^{\vartheta} \Psi(t, x[t])\, dt \tag{1.6}$$

Here $\varphi(t, x)$ and $\Psi(t, x)$ are given continuous functions which are defined for all x and for $t \in [t_0, T]$, where $T > t_0$ is some finite instant; $\vartheta = \vartheta(x[\cdot])$ is the time of initial arrival of the point $p[t]$ at the given closed set N. The first player, who is free to choose the control u, seeks to minimize γ (1.6); the second player seeks to maximize it. If the point $p[t]$ does not reach N for $t \in [t_0, T]$ we set $\gamma = \infty$ and assume that the first player has lost the game. The character of the information made available to the players is the same as in the previous problems. It is clear that if the set N is defined as a hyperplane $t = \vartheta = \text{const}$ in the vector space $p = \{t, x\}$ the games problem just described becomes a differential game with a fixed instant of termination $t = \vartheta$.

In solving the four game problems just stated we shall use the definitions and notation introduced in [1]. Let us state the results of [1] which form the basis of the discussion to follow.

Let $W(\tau, \vartheta)$ be the set of points w which satisfy the following condition: whatever the mixed second-player strategy $V = V(t, x)$ there exists a motion $x[t] = x[t; \tau, w, U_T, V]$ such that the point $p[t] = \{t, x[t]\}$ arrives at the set M not later than at the instant $t = \vartheta$ and the phase restriction $p[t] \in D$ is fulfilled for $t_0 \leqslant t \leqslant \vartheta(x[\cdot]; M)$. Here M and D are some closed sets in the vector space $p = \{t, x\}$; the symbol $\vartheta(x[\cdot]; M)$ denotes the instant when the condition $\{t, x[t]\} \in M$ is first fulfilled for some motion $x[t]$.

Theorem 1.1. If $x_0 \in W(t_0, \vartheta)$, then the mixed first-player strategy $U^{(e)} = U^{(e)}(t, x)$ extremal to the system of sets $W(t, \vartheta)$ $(t_0 \leqslant t \leqslant \vartheta)$ for any motion $x[t] = x[t; t_0, x_0, U^{(e)}, V_T]$ ensures fulfillment of the relations

$$\vartheta(x[\cdot]; M) \leqslant \vartheta, \qquad p[t] \in D \quad \text{for} \quad t_0 \leqslant t \leqslant \vartheta(x[\cdot]; M) \tag{1.7}$$

If $x_0 \notin W(t_0, \vartheta)$, then there exists an $\varepsilon > 0$ and a mixed second-player strategy $V_* = V_*(t, x)$ such that the following condition is fulfilled for any motion

$x[t] = x[t; t_0, x_0, U_T, V_*]$:

$$p[t] = \{t, x[t]\} \notin M^\varepsilon \quad \text{for} \quad t_0 \leqslant t \leqslant \min\{\vartheta, \tau^\varepsilon(x[\cdot]; D)\} \quad (1.8)$$

We recall that the symbol M^ε denotes the closed ε-neighborhood of the set M, and $\tau^\varepsilon(x[\cdot]; D)$ is the instant when the distance from the point $p[t] = \{t, x[t]\}$ to the set D is first equal to the number $\varepsilon > 0$.

Let $W_*(\tau, \vartheta)$ be the set of points w satisfying the following condition: whatever the mixed first-player strategy $U = U(t, x)$, there exists a motion $x[t] = x[t; \tau, w, U, V_T]$ for which the following relation holds:

$$p[t] = \{t, x[t]\} \in D \quad \text{for} \quad \tau \leqslant t \leqslant \min\{\vartheta, \vartheta(x[\cdot]; M)\}$$

Theorem 1.2. If $x_0 \in W_*(t_0, \vartheta)$, then the mixed second-player strategy $V^{(e)} = V^{(e)}(t, x)$ extremal to the system of sets $W_*(t, \vartheta)$ $(t_0 \leqslant t \leqslant \vartheta)$ ensures that the following condition is fulfilled for any motion $x[t] = x[t; t_0, x_0, U_T, V^{(e)}]$:

$$p[t] = \{t, x[t]\} \in D \quad \text{for} \quad t_0 \leqslant t \leqslant \min\{\vartheta, \vartheta(x[\cdot]; M)\} \quad (1.9)$$

If $x_0 \notin W_*(t_0, \vartheta)$, then there exists an $\varepsilon > 0$ and a mixed first-player strategy $U_* = U_*(t, x)$ such that the following conditions are fulfilled for any motion $x[t] = x[t; t_0, x_0, U_*, V_T]$:

$$\tau^\varepsilon(x[\cdot]; D) \leqslant \vartheta, \; p[t] = \{t, x[t]\} \notin M^\varepsilon \text{ for } t_0 \leqslant t \leqslant \tau^\varepsilon(x[\cdot]; D) \quad (1.10)$$

Note 1.1. We recall that Theorems 1.1 and 1.2 remain valid if we interchange the first and second players in the definitions of the sets $W(t, \vartheta)$ and $W_*(t, \vartheta)$ and in the statements of these theorems.

2. In this section we consider a differential guidance game with phase restriction, i.e. our Game Problem 1.1. Let $W^{(1)}(\tau, \vartheta)$ $(t_0 \leqslant \tau \leqslant \vartheta)$ be a set of points satisfying the following requirement: whatever the mixed second-player strategy $V = V(t, x)$, there exists a motion $x[t] = x[t; \tau, w, U_T, V]$ such that the following conditions are fulfilled:

$$\vartheta(x[\cdot]) = \vartheta(x[\cdot]; N) \leqslant \vartheta, \; \{t, x[t]\} \in G \quad \text{for} \quad \tau \leqslant t \leqslant \vartheta(x[\cdot])$$

We assume that there exists at least one value of the parameter ϑ for which the inclusion $x_0 \in W^{(1)}(t_0, \vartheta)$ holds. Otherwise (as will be shown below) there is no first-player strategy which guarantees completion of the differential guidance game with phase restriction in a finite time. We denote by ϑ° the number defined by the equation

$$\vartheta^\circ = \inf\{\vartheta : x_0 \in W^{(1)}(t_0, \vartheta)\} \quad (2.1)$$

i.e. ϑ° is the lower bound of all numbers ϑ for which the condition $x_0 \in W^{(1)}(t_0, \vartheta)$ holds. We can show that the following relation is valid:

$$x_0 \in W^{(1)}(t_0, \vartheta^\circ) \quad (2.2)$$

Clearly, the validity of inclusion (2.2) can be inferred from the following lemma.

Lemma 2.1. Let the number sequence $\vartheta^{(k)}$ and point sequence w_k satisfy the conditions

$$\vartheta^{(k)} \geqslant \vartheta^*, \quad w_k \in W^{(1)}(\tau, \vartheta^{(k)}) \quad (k = 1, 2, \ldots)$$
$$\lim_{k\to\infty} \vartheta^{(k)} = \vartheta^*, \quad \lim_{k\to\infty} w_k = w_*$$

The following inclusion then holds:

$$w_* \in W^{(1)}(\tau, \vartheta^*) \quad (2.3)$$

Proof. By the definition of the set $W^{(1)}(\tau, \vartheta^*)$ inclusion (2.3) must have the following meaning: whatever the mixed second-player strategy $V = V(t, x)$, there exists a motion $x_*[t] = x[t; \tau, w_*, U_T, V]$ such that

$$\vartheta(x_*[\cdot]) \leqslant \vartheta^*, \quad \{t, x_*[t]\} \in G \quad \text{for } \tau \leqslant t \leqslant \vartheta(x_*[\cdot]) \tag{2.4}$$

Thus, let $V = V(t, x)$ be an arbitrary mixed second-player strategy. Since $w_k \in W^{(1)}(\tau, \vartheta^{(k)})$, there exists a motion $x_k[t] = x[t; \tau, w_k, U_T, V]$ which satisfies the condition

$$\vartheta_k = \vartheta(x_k[\cdot]) \leqslant \vartheta^{(k)}, \quad p_k[t] = \{t, x_k[t]\} \in G \quad \text{for } \tau \leqslant t \leqslant \vartheta_k \tag{2.5}$$

From the number sequence ϑ_k and vector function sequence $p_k[t]$ we can choose convergent subsequences

$$\lim_{i\to\infty} \vartheta_{k_i} = \vartheta_* \leqslant \vartheta^*, \qquad \lim_{i\to\infty} (\max_{\tau\leqslant t\leqslant\vartheta_*} \| p_{k_i}[t] - p_*[t] \|) = 0 \tag{2.6}$$

$$p_{k_i}[t] = \{t, x_{k_i}[t]\}, \qquad p_*[t] = \{t, x_*[t]\}$$

We note that the vector function $x_*[t]$ is one of the motions $x[t; \tau, w_*, U_T, V]$ (see [1], Sect. 1 regarding the semicontinuity above with respect to inclusion of the system of motions $x[t; \tau, w, U_T, V]$ in the variable w). Since the sets N and G are closed, expressions (2.5) and (2.6) imply that the vector function $p_*[t] = \{t, x_*[t]\}$ satisfies the phase restriction $p_*[t] \in G$ for $\tau \leqslant t \leqslant \vartheta_*$; moreover, the inequality $\vartheta(x_*[\cdot]) \leqslant \leqslant \vartheta_* \leqslant \vartheta^*$ holds, i.e. condition (2.4) holds for the motion $x_*[t] = x[t; \tau, w_*, U_T, V]$. Lemma 2.1 has been proved.

We shall call the number ϑ°(2.1) the instant of positional absorption of the set N under the phase restriction $p \in G$. We have already shown that the instant of absorption ϑ° has the following property: for any $\vartheta < \vartheta^\circ$ we have $x_0 \notin W^{(1)}(t_0, \vartheta)$, and inclusion (2.2) is fulfilled for $\vartheta = \vartheta^\circ$. By virtue of Theorem 1.1, where we set $M = N$, $D = G$, inclusion (2.2) implies that the mixed first-player strategy $U^{(e)} = U^{(e)}(t, x)$ extremal to the system of sets $W^{(1)}(t, \vartheta^\circ)$ $(t_0 \leqslant t \leqslant \vartheta^\circ)$ ensures that the point $p[t] = \{t, x[t]\}$ arrives at N with fulfillment of the phase restriction $p[t] \in G$ not later than at the instant $t = \vartheta^\circ$. On the other hand, for $\vartheta < \vartheta^\circ$ we have $x_0 \notin W^{(1)}(t_0, \vartheta)$, so that (again by virtue of Theorem 1.1) we can say that there exists a mixed second-player strategy $V_* = V_*(t, x)$, which ensures fulfillment of condition (1.8). This implies that the following theorem is valid.

Theorem 2.1. In the game guidance problem with phase restriction whose payoff γ is defined by Eq. (1.3) there exists a game value wich is equal to $\vartheta^\circ - t_0$, where ϑ° is the instant of absorption defined by Eq. (2.1). Here the mixed first-player strategy extremal to the system of sets $W^{(1)}(t, \vartheta^\circ)$ $(t_0 \leqslant t \leqslant \vartheta^\circ)$ guarantees completion of the game with a payoff which satisfies the inequality $\gamma \leqslant \vartheta^\circ - t_0$, i.e. the strategy $U^{(e)}$ delivers the minimax of the payoff in the game under consideration. For any number $\vartheta < \vartheta^\circ$ there exists an $\varepsilon > 0$ and a mixed second-player strategy $V_* = V_*(t, x; \vartheta, \varepsilon)$ such that the following condition is fulfilled for any motion $x[t; t_0, x_0, U_T, V_*]$:

$$\{t, x[t]\} \notin N^\varepsilon \quad \text{for} \quad t_0 \leqslant t \leqslant \min\{\vartheta, \tau^\varepsilon(x[\cdot], G)\} \tag{2.7}$$

Thus, the strategy $V_* = V_*(t, x; \vartheta, \varepsilon)$ ensures completion of the game with a payoff which satisfies the inequality $\gamma \geqslant \vartheta - t_0$.

We note that as the strategy $V_* = V_*(t, x; \vartheta, \varepsilon)$ we can take a mixed second-player strategy extremal to the system of sets $W_*^{(1)}(\tau, \vartheta; \varepsilon)$ $(t_0 \leqslant \tau \leqslant \vartheta)$, where the sets $W_*^{(1)}(\tau, \vartheta; \varepsilon)$ are defined by the following condition: the point w belongs to the set $W_*^{(1)}(\tau, \vartheta; \varepsilon)$ if and only if for any mixed first-player strategy $U = U(t, x)$ there exists a motion $x[t] = x[t; \tau, w, U, V_T]$ for which the following relation is fulfilled:

$$\rho(\{t, x[t]\}; N) \geqslant \varepsilon \quad \text{for} \quad \tau \leqslant t \leqslant \min\{\vartheta, \tau^\varepsilon(x[\cdot]; G)\}$$

The symbol $\rho(p, N)$ denotes the distance from the point p to the set N.

Fulfillment of condition (2.7) for any motion

$$x[t] = x[t; t_0, x_0, U_T, V^{(e)}]$$

where $V^{(e)}$ is the mixed second-player strategy extremal to the system of sets $W_*^{(1)}(t, \vartheta; \varepsilon)$, follows from Theorem 1.2, in which the sets M and D are defined by the expressions

$$M = \{p : \rho(p, G) \geqslant \varepsilon\}, \qquad D = \{p : \rho(p, N) \geqslant \varepsilon\}$$

3. In this section we consider the solution of Game Problems 1.2 and 1.3, where the payoff is defined by Eqs. (1.4) and (1.5), respectively. Let us introduce some new notation. Let $H(c)$ and $K(c)$ be closed sets in the vector space $p = \{t, x\}$; these sets are defined as follows:

$$H(c) = \{p : \varphi(t, x) \geqslant c\}, \qquad K(c) = \{p : \varphi(t, x) \leqslant c\} \tag{3.1}$$

First let us turn to the solution of Problem 1.3. We introduce the sets $W^{(3)}(\tau, T; c)$ $(t_0 \leqslant \tau \leqslant T)$ consisting of all the points w which satisfy the following condition: for any mixed second-player strategy $V = V(t, x)$ there exists a motion $x[t] = x[t; \tau, w, U_T, V]$ for which the following relations are fulfilled:

$$\vartheta(x[\cdot]) = \vartheta(x[\cdot]; N)) \leqslant T, \quad \{t, x[t]\} \in K(c)$$
$$\text{for} \quad \tau \leqslant t \leqslant \vartheta(x[\cdot]) \tag{3.2}$$

We denote by c° the lower bound of all the numbers for which the inclusion

$$x_0 \in W^{(3)}(t_0, T; c) \tag{3.3}$$

holds.

Reasoning in the same way as we did in proving Lemma 2.1, we can show that the inclusion

$$x_0 \in W^{(3)}(t_0, T; c^\circ) \tag{3.4}$$

holds; here we must assume the existence of at least one $c < \infty$ for which condition (3.3) holds. Making use of Theorem 1.1, we can prove the following statement.

Theorem 3.1. Differential Game 1.3 in which the payoff is given by Eq. (1.5) has its value given by $c^\circ = \inf\{c : x_0 \in W^{(3)}(t_0, T; c)\}$. Here the mixed first-player strategy $U^{(e)} = U^{(e)}(t, x)$ extremal to the system of sets $W^{(3)}(t, T; c^\circ)$ $(t_0 \leqslant t \leqslant T)$ guarantees completion of the game with a payoff which satisfies the inequality $\gamma \leqslant c^\circ$; in other words, $U^{(e)}$ is the minimax strategy. For any number $c < c^\circ$ there exists an $\varepsilon > 0$ and a mixed second-player strategy $V_* = V_*(t, x; c, \varepsilon)$ such that for any motion $x[t] = x[t; t_0, x_0, U_T, V_*]$ the following condition is satisfied:

$$\rho(\{t, x[t]\}, N) \geqslant \varepsilon, \quad \text{for} \quad t_0 \leqslant t \leqslant \min\{T, \vartheta(x[\cdot]; H(c))\} \tag{3.5}$$

where $\vartheta(x[\cdot]; H(c))$ is the instant when the point $p[t] = \{t, x[t]\}$ arrives at the set $H(c)$ for the first time. Thus, the strategy V_* guarantees completion of the game with a payoff which satisfies the inequality $\gamma \geqslant c$.

Proof. Since inclusion (3.4) is valid, the definition of the system of sets $W^{(3)}(t, T; c^\circ)$ $(t_0 \leqslant t \leqslant T)$ and Theorem 1.1, where we set

$$M = N, \qquad D = K(c^\circ)$$

imply the following relations

$$\vartheta(x[\cdot]) \leqslant T, \quad \{t, x[t]\} \in K(c^\circ) \quad \text{for } t_0 \leqslant t \leqslant \vartheta(x[\cdot])$$

which are valid for any motion $x[t] = x[t; t_0, x_0, U^{(e)}, V_T]$.

The definition of the game payoff γ (1.5) and expressions (3.1) imply that the extremal strategy $U^{(e)}$ guarantees completion of the game with a payoff which satisfies the inequality $\gamma \leqslant c^\circ$. The first statement of the theorem has been proved.

Let $c < c^\circ$; then, by the definition of the number c° we have

$$x_0 \notin W^{(3)}(t_0, T; c) \tag{3.6}$$

Let us suppose that the second statement of the theorem is not valid, and let $V = V(t, x)$ be an arbitrary mixed second-player strategy. Since (3.5) does not hold, it follows that for any $\varepsilon > 0$ there exists a motion $x_\varepsilon[t] = x[t; t_0, x_0, U_T, V]$ and a number $t^*(\varepsilon)$ such that the following conditions are fulfilled:

$$t^*(\varepsilon) \leqslant T, \qquad \{t^*(\varepsilon), x_\varepsilon[t_\varepsilon^*]\} \in N^\varepsilon$$
$$\varphi(t, x_\varepsilon[t]) \leqslant c \quad \text{for} \quad t_0 \leqslant t \leqslant t^*(\varepsilon) \tag{3.7}$$

Making use of the property of compactness in itself of the set of motions $x[t; t_0, x_0, U_T, V]$ (see [1], Sect. 1) and taking the limit as $\varepsilon \to 0$, we find from (3.7) that there exists a motion $x[t] = x[t; t_0, x_0, U_T, V]$ which satisfies conditions (3.2) for $\tau = t_0$. Since $V = V(t, x)$ is an arbitrary second-player strategy, the inclusion $x_0 \in W^{(3)}(t_0, T; c)$ holds for the point x_0. But this contradicts relation (3.6), which in turn proves the validity of the second statement of the theorem. Theorem 3.1 has been proved.

We note that the strategy $V_* = V_*(t, x; c, \varepsilon)$ can be defined as extremal to the system of sets $W_*^{(3)}(t, T; c, \varepsilon)$ $(t_0 \leqslant t \leqslant T)$. Here the sets $W_*^{(3)}(\tau, T; c, \varepsilon)$ must be defined as the collection of points w which satisfy the following condition: for any mixed first-player strategy $U = U(t, x)$ there exists a motion $x[t] = x[t; \tau, w, U, V_T]$ for which the following condition is satisfied:

$$\rho(\{t, x[t]\}, N) \geqslant \varepsilon \quad \text{for} \quad \tau \leqslant t \leqslant \min\{T, \vartheta(x[\cdot]; H(c))\}$$

Fulfillment of condition (3.5) for any motion

$$x[t] = x[t; t_0, x_0, U_T, V^{(e)}]$$

where $V^{(e)}$ is a mixed second player strategy extremal to the system of sets $W_*^{(3)}(t, T; c, \varepsilon)$, follows from Theorem 1.2, where we set $M = H(c)$, $D = \{p: \rho(p, N) \geqslant \varepsilon\}$.

Now let us turn to Problem 1.2. It is easy to see that Game Problem 1.2 coincides with Differential Game 1.3 if we add the hyperplane $t = T$ to the set N (the arrival of the point $p[t]$ at this set coincides with the instant of termination of Game 1.3). Then for all motions $x[t]$ we have $\vartheta^*(x[\cdot]) \leqslant T$, so that payoff (1.4) coincides with payoff (1.5). Thus, Differential Game 1.2 is a particular case of Game Problem 1.3 considered above. We therefore omit discussing this game in detail, and merely state the final result.

Let $W_*^{(2)}(\tau, T; c)$ be the collection of points w which satisfy the following condition: for any mixed second-player strategy $V = V(t, x)$ there exists a motion $x[t] = x[t; \tau, w, U_T, V]$ for which the relation $\{t, x[t]\} \in K(c)$ holds for $\tau \leqslant t \leqslant \min\{T, \vartheta(x[\cdot])\}$.

T h e o r e m 3.2. Differential Game 1.2 whose payoff is given by Eq. (1.4) has its value given by

$$c_*^\circ = \inf \{c;\ x_0 \in W_*^{(2)}(t_0,\ T;\ c)\}$$

Here the mixed first-player strategy $U^{(e)} = U^{(e)}(t,\ x)$ extremal to the system of sets $W_*^{(2)}(t,\ T;\ c^\circ)\ (t_0 \leqslant t \leqslant T)$ guarantees completion of the game with a payoff satisfying the inequality $\gamma \leqslant c_*^\circ$; in other words, $U^{(e)}$ is a minimax strategy. For any number $c < c_*^\circ$ there exists a $\varepsilon > 0$ and a mixed second-player strategy $V_* = V_*(t,\ x;\ c,\ \varepsilon)$ such that the following relations are fulfilled for any motion $x[t] = x[t;\ t_0,\ x_0,\ U_T,\ V_*]$:

$$\vartheta(x[\cdot];\ H(c)) \leqslant T, \quad \rho(\{t,\ x[t]\},\ N) \geqslant \varepsilon \quad \text{for} \quad t_0 \leqslant t \leqslant \vartheta(x[\cdot];\ H(c))$$

Thus, the strategy V_* guarantees completion of the game with a payoff which satisfies the inequality $\gamma \geqslant c$.

We note that here too the strategy V_* can be defined as extremal to the system of sets $W^{(2)}(t,\ T;\ c,\ \varepsilon)\ (t_0 \leqslant t \leqslant T)$. Here the sets $W^{(2)}(\tau,\ T;\ c,\ \varepsilon)$ must be defined as the collection of points w which satisfy the following requirement: whatever the mixed first-player strategy $U = U(t,\ x)$, there exists a motion $x[t] = x[t;\ \tau,\ w,\ U,\ V_T]$ such that the conditions

$$\vartheta(x[\cdot];\ H(c)) \leqslant T, \quad \rho(\{t,\ x[t]\},\ N) \geqslant \varepsilon \quad \text{for} \quad t_0 \leqslant t \leqslant \vartheta(x[\cdot];\ H(c))$$

are fulfilled.

4. Finally, let us consider Differential Game 1.4. This differential game can be replaced in familiar fashion by the equivalent differential game of the same form as 1.4 whose payoff does not contain the second term. To this end we need merely add the new component ξ to the phase coordinates of system (1.1); the variation of this new component is defined by the equation

$$d\xi\ /\ dt = \psi(t,\ x), \qquad \xi[t_0] = 0 \tag{4.1}$$

In the differential game equivalent to the initial Game Problem 1.4 the payoff is given by the equation

$$\gamma = \varphi^*(\vartheta,\ x[\vartheta],\ \xi[\vartheta]) \tag{4.2}$$

where $\varphi^*(t,\ x,\ \xi) = \varphi(t,\ x) + \xi,\ \vartheta = \vartheta(x[\cdot])$ is, as before, the instant of arrival of the point $p[t] = \{t,\ x[t]\}$ at the set N. From now on (to simplify our expressions and notation) we shall assume that the required transformations have been effected, and that the payoff of the game under consideration is of the form

$$\gamma = \varphi(\vartheta,\ x[\vartheta]) \quad (\vartheta = \vartheta(x[\cdot]) \leqslant T) \tag{4.3}$$

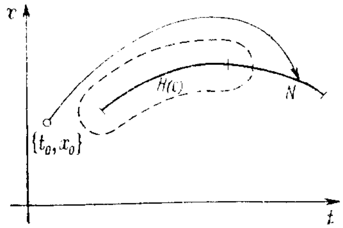

Fig. 1

Let us introduce some of the symbols to be used below. Let $H(c)$ and $K(c)$ be closed sets in the vector space p; these sets are defined by the relations

$$H(c) = \{p : p \in N, \quad \varphi(t,\ x) \geqslant c\}$$
$$K(c) = \{p : p \in N, \quad \varphi(t,\ x) \leqslant c\} \tag{4.4}$$

By $C^\circ(\beta)$ we denote the set of all numbers c which satisfy the following condition.

Condition 4.1. Whatever the mixed second-player strategy $V = V(t,\ x)$, there exists a motion $x[t] = x[t;\ t_0,\ x_0,\ U_T,\ V]$ for which the following relations hold:

$$\vartheta(x[\cdot]) \leqslant T,\ \rho(\{t, x[t]\}, H(c)) \geqslant \beta > 0 \quad \text{for } t_0 \leqslant t \leqslant \vartheta(x[\cdot]) \qquad (4.5)$$

Figure 1 shows the sets N, $H(c)$ and a motion $x[t]$ which satisfies condition (4.5). The broken curve indicates the boundary of the set $H^\beta(c)$.

We denote by c° the number defined by the equation

$$c^\circ = \inf c \quad \text{for } c \in \bigcup C^\circ(\beta) \qquad (\beta > 0) \qquad (4.6)$$

The following statement is valid.

Theorem 4.1. If $c > c^\circ$, then there exists a $\beta > 0$ and a mixed first-player strategy $U_0 = U_0(t, x; c, \beta)$ such that the following conditions hold for any motion $x[t] = x[t; t_0, x_0, U_0, V_T]$:

$$\vartheta(x[\cdot]) \leqslant T, \qquad \rho(\{t, x[t]\}, H(c)) \geqslant \beta \qquad \text{for} \qquad t_0 \leqslant t \leqslant \vartheta(x[\cdot]) \qquad (4.7)$$

i.e. the strategy U_0 ensures completion of the game with a payoff satisfying the inequality $\gamma < c$. On the other hand, if $c < c^\circ$, then for any arbitrarily small $\beta > 0$ there exists a mixed second-player strategy $V_* = V_*(t, x; c, \beta)$ which ensures fulfillment of at least one of the following two relations:

$$\rho(\{t, x[t]\}, N) > 0 \quad \text{for} \quad t \in [t_0, T] \qquad (4.8)$$
$$\min \rho(\{t, x[t]\}, H(c)) \leqslant \beta \quad \text{for} \quad t_0 \leqslant t \leqslant \vartheta(x[\cdot])$$

for any motion $x[t] = x[t; t_0, x_0, U_T, V_*]$.

Proof. Let $c > c^\circ$. By the definition of the number c° this inequality implies the existence of a $\beta > 0$ and of a c^* from the set $C^\circ(\beta)$ such that the inequality $c > c^*$ is fulfilled. Let us introduce the system of sets $W^{(4)}(\tau, T; c^*, \beta)$ $(t_0 \leqslant \tau \leqslant T)$. We define the system of sets $W^{(4)}(\tau, T; c^*, \beta)$ as the collection of all points w for which the following condition holds: for any mixed second-player strategy $V = V(t, x)$ there exists a motion $x[t] = x[t; \tau, w, U_T, V]$ which satisfies the relations

$$\vartheta(x[\cdot]) \leqslant T, \qquad \rho(\{t, x[t]\}, H(c^*)) \geqslant \beta \quad \text{for } \tau \leqslant t \leqslant \vartheta(x[\cdot])$$

Since the number c^* belongs to the set $C^\circ(\beta)$, the definitions of the sets $C^\circ(\beta)$ and $W^{(4)}(\tau, T; c^*, \beta)$ imply the validity of the inclusion

$$x_0 \in W^{(4)}(t_0, T; c^*, \beta) \qquad (4.9)$$

Now let us define $U_0 = U_0(t, x; c, \beta)$ as the strategy extremal to the system of sets $W^{(4)}(t, T; c^*, \beta)$. Fulfillment of relations (4.7) then follows from Theorem 1.1 where we set $M = N$, $D = \{p\colon \rho(p, H(c^*)) \geqslant \beta\}$, and also from relation (4.9) and the inequality $c > c^*$. The validity of the inequality $\gamma < c$ follows directly from the definition of payoff (4.3) and of the set $H(c)$ (4.4). The first statement of Theorem 4.1 has been proved.

Now let $c < c^\circ$. Then, whatever the number $\beta > 0$, the number c does not belong to the set $C^\circ(\beta)$. This means (by the definition of the set $C^\circ(\beta)$) that there exists a strategy V_* which ensures fulfillment of one of the relations of (4.8). This means that the second statement of Theorem 4.1 is also valid.

Thus, Theorem 4.1 provides an estimate of the game payoff which the first player can ensure on his own behalf by choosing mixed strategies $U_0 = U_0(t, x; c, \beta)$ extremal to the system of sets $W^{(4)}(t, T; c, \beta)$. Let us now obtain a similar estimate of the guaranteed payoff for the second player.

We denote by $C_0(\beta)$ the set of numbers c for which the following condition is fulfilled.

Condition 4.2. Whatever the mixed first-player strategy $U = U(t, x)$, there exists a motion $x[t] = x[t; t_0, x_0, U, V_T]$ which satisfies the following relation:

$$\rho(\{t, x[t]\}, K(c)) \geqslant \beta > 0 \quad \text{for} \quad t_0 \leqslant t \leqslant \min\{T, \vartheta(x[\cdot]\}$$

Now let c_0 be a number defined by the equation

$$c_0 = \sup c \quad \text{for} \quad c \in \bigcup C_0(\beta) \; (\beta > 0) \tag{4.10}$$

Theorem 4.2. If $c < c_0$, then there exists a $\beta > 0$ and a mixed second-player strategy $V_0 = V_0(t, x; c, \beta)$ such that the following relation is fulfilled for any motion $x[t] = x[t; t_0, x_0, U_T, V_0]$:

$$\rho(\{t, x[t]\}, K(c)) \geqslant \beta \quad \text{for} \quad t_0 \leqslant t \leqslant \min\{T, \vartheta(x[\cdot])\} \tag{4.11}$$

In other words, the strategy V_0 ensures that the second player completes the game with a payoff which satisfies the inequality $\gamma > c$. On the other hand, if $c > c_0$, then for any positive number $\beta > 0$ there exists a mixed first-player strategy $U_* = U_*(t, x; c, \beta)$ which ensures fulfillment of the following condition for any motion $x[t] = x[t; t_0, x_0, U_*, V_T]$:

$$\min \rho(\{t, x[t]\}, K(c)) \leqslant \beta \quad \text{for} \quad t_0 \leqslant t \leqslant \min\{T, \vartheta(x[\cdot])\} \tag{4.12}$$

The proof of this theorem is similar to the proof of Theorem 4.1. The sole difference is that instead of the system of sets $W^{(4)}(\tau, T; c, \beta)$ $(t_0 \leqslant \tau \leqslant T)$ we must introduce the sets $W_*^{(4)}(\tau, T; c, \beta)$ defined as the collections of all points w which satisfy the following requirement : for any mixed first-player strategy $U = U(t, x)$ there exists a motion $x[t] = x[t; \tau, w, U, V_T]$ for which the condition

$$\rho(\{t, x[t]\}, K(c)) \geqslant \beta \quad \text{for} \quad \tau \leqslant t \leqslant \min\{T, \vartheta(x[\cdot])\}$$

holds.

Since $c < c_0$, there exists a $\beta > 0$ such that

$$x_0 \in W_*^{(4)}(t_0, T; c, \beta) \tag{4.13}$$

Assuming that $V_0 = V_0(t, x; c, \beta)$ is a second-player strategy extremal to the system of sets $W_*^{(4)}(t, T; c, \beta)$ $(t_0 \leqslant t \leqslant T)$, we find that the validity of (4.11) follows from condition (4.13) and Theorem 1.2, where we set $M = N, D = \{p: \rho(p, K(c)) \geqslant \beta\}$.

For any $\beta > 0$ the number c does not belong to the set $C_0(\beta)$ if $c > c_0$, so that the second statement of Theorem 4.2 follows directly from the definition of the sets $C_0(\beta)$.

Thus, if the numbers c^0 and c_0 are defined by relations (4.6), (4.10), then the first player can ensure that he will complete the game with a payoff satisfying the inequality $\gamma \leqslant c^\circ + \varepsilon$, and the second player can ensure fulfillment of the inequality $\gamma \geqslant c_0 - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. It is easy to see that the numbers c° and c_0 are related by the inequality $c_0 \leqslant c^\circ$. We can show that the numbers c_0 and c° coincide under certain additional conditions. Specifically, as we know from Sect. 2, the differential guidance game has value; in addition, as is shown below, this game can be reduced to a differential game in which the payoff is defined by Eq. (4.3). It is therefore interesting to determine the additional conditions which are fulfilled for this problem and ensure that the numbers c_0 and c° coincide. We state such a condition and then prove it for the differential guidance game.

Condition 4.3. For any motion $x[t] = x[t; t_0, x_0, U_T, V_T]$, fulfillment of the relation $\{t_*, x[t_*]\} \in K(c)$, where $t_* \leqslant T$, the following condition must be satisfied:

$$\{\vartheta(x[\cdot]), x[\vartheta(x[\cdot])]\} \in K(c) \tag{4.14}$$

In addition, for any $\alpha > 0$ there exists a $\beta > 0$ such that when

$$\{\vartheta(x[\cdot]), x[\vartheta(x[\cdot])]\} \in K(c), \qquad \vartheta(x[\cdot]) \leqslant T \tag{4.15}$$

we have the inequality

$$\rho(\{t, x[t]\}, H(c+\alpha)) \geqslant \beta \quad \text{for} \quad t_0 \leqslant t \leqslant \vartheta(x[\cdot]) \tag{4.16}$$

for any motion $x[t] = x[t; t_0, x_0, U_T, V_T]$ satisfying relation (4.15).

Theorem 4.3. If Condition 4.3 is fulfilled, then $c_0 = c^\circ$.

Proof. Let us suppose that this is not the case. Let the numbers c and α satisfy the inequalities $c_0 < c < c+\alpha < c^\circ$. By virtue of the second statement of Theorem 4.2, for any arbitrarily small $\beta > 0$ we have

$$\rho(\{t_*, x[t_*]\}, K(c)) \leqslant \beta \quad \text{for} \quad t_0 \leqslant t_* \leqslant \min\{T, \vartheta(x[\cdot])\}$$

for any motion $x[t] = x[t; t_0, x_0, U_*, V]$, where $V = V(t, x)$ is some mixed second-player strategy. Choosing from among the motions $x[t] = x[t; t_0, x_0, U_*, V]$ corresponding to distinct values of the parameter $\beta > 0$ some sequence which converges as $\beta \to 0$ whose limit we denote by $x^*[t]$ $(t_0 \leqslant t \leqslant T)$, we find that $\{t^*, x^*[t^*]\} \in K(c)$, where t^* is some number from the interval $[t_0, T]$. We note that $x^*[t]$ $(t_0 \leqslant t \leqslant T)$ coincides with one of the motions $x[t; t_0, x_0, U_T, V]$ and by virtue of condition (4.14) we have the inclusion $\{\vartheta(x^*[\cdot]), x^*[\vartheta(x^*[\cdot])]\} \in K(c)$, which by virtue of (4.15), (4.16) implies the relation

$$\rho(\{t, x^*[t]\}, H(c+\alpha)) \geqslant \beta > 0 \quad \text{for} \quad t_0 \leqslant t \leqslant \vartheta(x^*[\cdot]) \leqslant T$$

Since $V = V(t, x)$ is an arbitrary mixed second-player strategy, and since here the number $\beta > 0$ does not depend on V, we can state that the number $c+\alpha$ belongs to the set $C^\circ(\beta)$, so that $c+\alpha \geqslant c^\circ$; but this contradicts the assumption that $c+\alpha < c^\circ$. This contradiction proves the validity of Theorem 4.3.

Setting $\varphi \equiv 0$, $\Psi \equiv 1$ in (1.6), we obtain a differential guidance game which differs from the differential game considered in Sect. 2 in the fact that it is not subject to the phase restriction $\{t, x[t]\} \in G$. However, this distinction makes no difference in the discussion to follow. (Game Problem 1.4 can be stated in such a way as to require fulfillment of the restriction $p[t] \in G$ from the first player; Problems 1.1 and 1.4 then coincide completely for $\varphi \equiv 0$, $\Psi \equiv 1$. Here the introduction of the additional restriction $p[t] \in G$ does not significantly alter the proofs of the statements made in the present section). Let us show that Condition 4.3 is fulfilled for $\varphi \equiv 0$, $\Psi \equiv 1$, i.e. for the guidance problem.

In fact, the sets $H(c)$ and $K(c)$, which must be constructed in the $(n+2)$-dimentional vector space $\{t, x, \xi\}$, are defined by the conditions

$$H(c) = \{p = \{t, x\} \in N, \ \xi = t - t_0 \geqslant c\}, \quad K(c) = \{p = \{t, x\} \in N, \ \xi = t - t_0 \leqslant c\}$$

It is clear that the condition $\{t_*, x[t_*], \xi[t_*]\} \in K(c)$ implies the validity of the inclusion $\{\vartheta(x[\cdot]), x[\vartheta(x[\cdot])], \xi[\vartheta(x[\cdot])]\} \in K(c)$, since $\vartheta(x[\cdot]) \leqslant t_*$. It is equally easy to verify the validity of inequality (4.16) if condition (4.15) is fulfilled. Thus, Condition 4.3 holds in this case.

Theorem 4.3 implies that Differential Game 1.4 has a value. This fact, which we established in Sect. 2, was inferred from the material presented in this section. We note that Condition 4.3 is valid in the general case, i.e. for Differential Game 1.4 in which $\varphi(t, x) \equiv 0$, $\Psi(t, x) > 0$ for all $\{t, x\}$. In addition, Condition 4.3 is also fulfilled for Differential Game 1.4 when its instant of termination has been specified, i.e. in relation (1.6) defining the payoff $\vartheta = T = \text{const}$. Thus, Differential Game 1.4 with a fixed instant of termination also has value. The material of the present section clearly implies that the optimal strategies in this game are the player strategies extremal to the appropriately defined absorption sets $W^{(4)}(t, T)$ and $W_*^{(4)}(t, T)$ $(t_0 \leqslant t \leqslant T)$.

In conclusion, we cite an example in which the numbers c° (4.6) and c_0 (4.10) do not coincide.

Let the system motion be described by the equations

$$dx_1/dt = b_1(t)v, \qquad x_1[0] = 0$$
$$dx_2/dt = b_2(t)u, \qquad x_2[0] = 0$$

where $b_1(t)$ and $b_2(t)$ are some continuous functions satisfying the conditions

$$b_1(t) > 0 \quad \text{for} \quad 0 \leqslant t < 1, \qquad b_1(t) = 0 \quad \text{for} \quad t \geqslant 1$$
$$b_2(t) = 0 \quad \text{for} \quad 0 \leqslant t \leqslant 1, \qquad b_2(t) > 0 \quad \text{for} \quad t > 1$$

The controls u and v are subject to the restrictions

$$0 \leqslant u \leqslant 1, \quad -1 \leqslant v \leqslant 0$$

The set N consists of two points, namely

$$A_1 = \{t = 1,\ x_1 = 0,\ x_2 = 0\}, \qquad A_2 = \{t = 2,\ x_1 = 0,\ x_2 = 0\}$$

the two straight lines

$$L_1 = \{t = 3,\ x_2 = 0\}, \qquad L_2 = \{t = 4,\ x_2 = 0\}$$

and the plane

$$\Gamma = \{p = \{t, x_1, x_2\} : t = 5\}$$

The function $\varphi(t, x_1, x_2)$ on the set N is defined as follows: at the point A_1 and at the straight line L_2 we set $\varphi(t, x_1, x_2) = 0$; at the point A_2, at the straight line L_1, and at the plane Γ we set $\varphi(t, x_1, x_2) = 1$.

The definitions of the numbers c° (4.6) and c_0 (4.10) imply directly that in this example $c^\circ = 1$, $c_0 = 0$.

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Transalted by A. Y.